

AVERAGES OF TWISTED L -FUNCTIONS

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ABSTRACT. We use a relative trace formula on $\mathrm{GL}(2)$ to compute a sum of twisted modular L -functions anywhere in the critical strip, weighted by a Fourier coefficient and a Hecke eigenvalue. When the weight k or level N is sufficiently large, the sum is nonzero. Specializing to the central point, we show in some cases that the resulting bound for the average is as good as that predicted by the Lindelöf hypothesis in the k and N aspects.

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1. INTRODUCTION

Let $S_k(N, \psi)$ be the space of cusp forms h on $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \mid c \in N\mathbf{Z} \right\}$ satisfying

$$h\left(\frac{az+b}{cz+d}\right) = \psi(d)(cz+d)^k h(z)$$

for all z in the complex upper half-plane \mathbf{H} and all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, where ψ is a Dirichlet character modulo N . We normalize the Petersson inner product on $S_k(N, \psi)$ by

$$(1.1) \quad \|h\|^2 = \frac{1}{\nu(N)} \iint_{\Gamma_0(N) \backslash \mathbf{H}} |h(z)|^2 y^k \frac{dx dy}{y^2},$$

where

$$\nu(N) = [\mathrm{SL}_2(\mathbf{Z}) : \Gamma_0(N)].$$

Given $h \in S_k(N, \psi)$, write $h(z) = \sum_{n>0} a_n(h) q^n$ for $q = e^{2\pi iz}$. Fix an integer D with $(D, N) = 1$, and let χ be a primitive Dirichlet character modulo D . The χ -twisted L -function of h is given for $\mathrm{Re}(s) > \frac{k}{2} + 1$ by the Dirichlet series

$$L(s, h, \chi) = \sum_{n>0} \frac{\chi(n) a_n(h)}{n^s}.$$

The completed L -function

$$\Lambda(s, h, \chi) = (2\pi)^{-s} \Gamma(s) L(s, h, \chi)$$

has an analytic continuation to the complex plane and satisfies a functional equation relating s to $k-s$, so the central point is $s = k/2$. Taking χ trivial and $D = 1$ gives the usual L -function $\Lambda(s, h)$. When $N = 1$, the functional equation takes the form

$$(1.2) \quad \Lambda(s, h, \chi) = \frac{i^k}{D^{2s-k}} \frac{\tau(\chi)^2}{D} \Lambda(k-s, h, \overline{\chi}),$$

where

$$(1.3) \quad \tau(\chi) = \sum_{m \in (\mathbf{Z}/D\mathbf{Z})^*} \chi(m) e^{2\pi i m/D}$$

is the Gauss sum attached to χ .

Let n be an integer prime to N , and let T_n be the n -th Hecke operator, given by

$$T_n h(z) = n^{k-1} \sum_{\substack{ad=n, \\ a>0}} \sum_{b=0}^{d-1} \psi(a) d^{-k} h\left(\frac{az+b}{d}\right).$$

Let \mathcal{F} be an orthogonal basis for $S_k(N, \psi)$ consisting of eigenfunctions of T_n . We denote the Hecke eigenvalue by $T_n h = \lambda_n(h)h$, and recall that $a_n(h) = a_1(h)\lambda_n(h)$. Our main result is the following.

Theorem 1.1. *With notation as above, assume $k > 2$, and let $r, n \in \mathbf{Z}^+$ with $(rn, D) = 1$. Then for all $s = \sigma + i\tau$ in the strip $1 < \sigma < k-1$,*

$$(1.4) \quad \begin{aligned} & \frac{1}{\nu(N)} \sum_{h \in \mathcal{F}} \frac{\lambda_n(h) \overline{a_r(h)} \Lambda(s, h, \chi)}{\|h\|^2} \\ &= \frac{2^{k-1} (2\pi rn)^{k-s-1}}{(k-2)!} \Gamma(s) \sum_{d|(n,r)} d^{2s-k+1} \psi\left(\frac{n}{d}\right) \chi\left(\frac{rn}{d^2}\right) \\ & \quad + \delta_{N,1} \frac{2^{k-1} (2\pi rn)^{s-1}}{(k-2)!} \Gamma(k-s) \frac{i^k}{D^{2s-k}} \frac{\tau(\chi)^2}{D} \sum_{d|(r,n)} d^{k-2s+1} \overline{\chi\left(\frac{rn}{d^2}\right)} \\ & \quad + E, \end{aligned}$$

where $\delta_{N,1} \in \{0, 1\}$ is nonzero iff $N = 1$, and the error term E is an infinite series involving confluent hypergeometric functions (cf. Proposition 8.1) satisfying

$$(1.5) \quad |E| \leq 2 \gcd(r, n) \frac{(4\pi rn)^{k-1} D^{k-\sigma-\frac{1}{2}} \varphi(D) B(\sigma, k-\sigma)}{N^\sigma (k-2)!} \cosh\left(\frac{\pi\tau}{2}\right) \zeta(k-\sigma) \zeta(\sigma).$$

Here $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is Euler's Beta function and $\varphi(D)$ is Euler's φ -function.

Theorem 1.1 extends first moment calculations of many authors, including [Du], [Ak], and [Ka]. Rather than using the Petersson formula as in these references, here we compute the average directly with the relative trace formula. This method was also used in [KL2], which treated the untwisted case. An immediate application is the nonvanishing of L -functions (see §9).

Corollary 1.2. *Suppose $N > 1$ and $\gcd(r, n) = 1$. Then for any s in the critical strip $\frac{k-1}{2} < \operatorname{Re}(s) < \frac{k+1}{2}$, the sum (1.4) is nonzero as long as $N+k$ is sufficiently large.*

This can be interpreted as a GRH-on-average for the twisted L -functions. When $N = 1$, we cannot prove nonvanishing on the critical line $\operatorname{Re}(s) = \frac{k}{2}$, since the first two terms have the same magnitude there. Indeed, they cancel out at $s = \frac{k}{2}$ if χ is quadratic and conditions on k, D conspire in (1.2) to force the L -functions to vanish. However, by arguments given in [KL2] one can show that when $N = 1$ and $\operatorname{Re}(s) \neq \frac{k}{2}$, the sum (1.4) is nonzero if k is sufficiently large.

According to the generalized Lindelöf hypothesis, for a newform h we have

$$(1.6) \quad L\left(\frac{k}{2}, h, \chi\right) \ll (D^2 N k)^\varepsilon.$$

Let $\mathcal{F}_k(N)^{\text{new}}$ be any orthogonal basis for the span $S_k(N)^{\text{new}}$ of the newforms with trivial character. Using the fact ([Ser], p. 86) that $\dim S_k(N)^{\text{new}} \sim \frac{k-1}{12} \nu(N)^{\text{new}}$

where $N^{1-\varepsilon} \ll \nu(N)^{\text{new}} \leq N$, (1.6) implies the following averaged Lindelöf hypothesis:

$$(1.7) \quad \sum_{h \in \mathcal{F}_k(N)^{\text{new}}} L\left(\frac{k}{2}, h, \chi\right) \ll D^\varepsilon (Nk)^{1+\varepsilon}.$$

One can use Theorem 1.1 to prove certain instances of (1.7) unconditionally, although from (1.5) it is clear that we cannot achieve adequate bounds in the D -aspect. The idea is to set $n = r = 1$ in (1.4) and use well-known bounds for the Petersson norm, along with positivity of the L -values when χ is real. If oldforms are present, the method apparently grinds to a halt because $\frac{a_1(h)\Lambda(\frac{k}{2}, h, \chi)}{\|h\|^2}$ may be negative. Indeed, even in the simplest case where $N = p$ is prime, if h is a newform of level 1, and h_p is a nonzero basis element (unique up to scaling) orthogonal to h in $\text{Span}\{h(z), h(pz)\}$, then using [ILS] (2.45)-(2.46) it is not hard to show that

$$(1.8) \quad \overline{a_1(h_p)}\Lambda\left(\frac{k}{2}, h_p, \chi\right) = \mu \left(\frac{\lambda_p(h)^2}{(p+1)^2} - \frac{\lambda_p(h)\chi(p)}{p^{1/2}(p+1)} \right) \Lambda\left(\frac{k}{2}, h, \chi\right)$$

for some constant $\mu > 0$ depending on the choice of h_p .¹ The above can clearly be negative, for example if $\chi(p) = 1$ and the real eigenvalue $\lambda_p(h)$ is close to 1. Therefore we have to content ourselves here with cases in which oldforms are not present. We highlight two such cases, one in the k -aspect and one in the N aspect, though the proof applies more generally.

Corollary 1.3. *Let $\mathcal{F}_k(1)$ be an orthogonal basis for $S_k(1)$ consisting of newforms, normalized with first Fourier coefficient equal to 1. Then for any real primitive Dirichlet character χ ,*

$$(1.9) \quad \sum_{h \in \mathcal{F}_k(1)} L\left(\frac{k}{2}, h, \chi\right) \ll_{\varepsilon, D} k^{1+\varepsilon}.$$

Let $4 \leq k_0 \leq 14$ be an even integer not equal to 12, let N be a prime not dividing the conductor of χ , and let $\mathcal{F}_{k_0}(N)^{\text{new}}$ be an orthogonal basis for $S_{k_0}(N)^{\text{new}} = S_{k_0}(N)$ consisting of normalized newforms. Then

$$(1.10) \quad \sum_{h \in \mathcal{F}_{k_0}(N)^{\text{new}}} L\left(\frac{k_0}{2}, h, \chi\right) \ll_{\varepsilon, D} N^{1+\varepsilon}.$$

Remark: The estimate (1.9) was first proven by Kohnen and Sengupta using Waldspurger's formula relating the central values to certain Fourier coefficients of half-integral weight modular forms ([KS]). The case of trivial χ was proven earlier by Sengupta by essentially the same method we use here ([Sen]).

Proof. In Theorem 1.1, suppose that the central character ω' is trivial and that χ is real. We assume that there are no oldforms, so \mathcal{F} can be chosen to consist of newforms h , normalized with $a_1(h) = 1$. By the hypotheses on ω' and χ , we have $L(\frac{k}{2}, h, \chi) \geq 0$ for all $h \in \mathcal{F}$ ([Gu]). Furthermore, we have the bound

$$\frac{(4\pi)^{k-1}}{(k-2)!} \nu(N) \|h\|^2 \ll (kN)^{1+\varepsilon}$$

¹ By [ILS] (2.45), $h_p = w \left(-\frac{\lambda_p(h)}{p+1} h(z) + p^{\frac{k-1}{2}} h(pz) \right)$ for some nonzero $w \in \mathbf{C}$; by (2.46), $a_1(h_p) = -w \frac{\lambda_h(p)}{p+1}$; by an easy manipulation, $L(\frac{k}{2}, h(pz), \chi) = \frac{\chi(p)}{p^s} L(s, h, \chi)$, so $L(s, h_p, \chi) = w \left(-\frac{\lambda_p(h)}{p+1} + \frac{\chi(p)}{p^{1/2}} \right) L(\frac{k}{2}, h, \chi)$. Then (1.8) holds with $\mu = |w|^2$.

for all newforms $h \in \mathcal{F}$ (see (2.29) of [IM]). Therefore due to the nonnegativity of the L -values, we have

$$\begin{aligned} \sum_{h \in \mathcal{F}} L(\tfrac{k}{2}, h, \chi) &\ll \frac{(kN)^{1+\varepsilon}(k-2)!}{(4\pi)^{k-1}\nu(N)} \sum_{h \in \mathcal{F}} \frac{L(\tfrac{k}{2}, h, \chi)}{\|h\|^2} \\ &= \frac{(kN)^{1+\varepsilon}(k-2)!}{2^{k-1}(2\pi)^{k/2-1}\Gamma(\tfrac{k}{2})\nu(N)} \sum_{h \in \mathcal{F}} \frac{\Lambda(\tfrac{k}{2}, h, \chi)}{\|h\|^2}. \end{aligned}$$

Applying the theorem with $n = r = 1$, we immediately obtain

$$\sum_{h \in \mathcal{F}} L(\tfrac{k}{2}, h, \chi) \ll (kN)^{1+\varepsilon} \left(1 + \delta_{N,1} \frac{i^k \tau(\chi)^2}{D} + \frac{(k-2)!}{2^{k-1}(2\pi)^{k/2-1}\Gamma(\tfrac{k}{2})} E \right).$$

It is clear from (1.5) that the third term in the parentheses tends to 0 as $N \rightarrow \infty$. Using Stirling's approximation, it is not hard to show that the same is true as $k \rightarrow \infty$ (see §9 for details), and the corollary follows. \square

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2. NOTATION AND PRELIMINARIES

Let \mathbf{A} , \mathbf{A}_{fin} be the adeles and finite adeles of \mathbf{Q} , respectively. Fix a positive integer N . For $x \in \mathbf{A}^*$, we let x_N denote the idele whose p -th component is x_p for all $p|N$ and 1 for all $p \nmid N$. For any integer d , we also write $d_p = \text{ord}_p(d)$ (the p -adic valuation of d). It should be clear from the context which meaning we take when a subscript p appears.

Let ψ be a Dirichlet character modulo N , extended to \mathbf{Z} by $\psi(d) = 0$ if $(d, N) > 1$. We let ψ^* denote its adelic counterpart (a Hecke character), defined via strong approximation $\mathbf{A}^* = \mathbf{Q}^*(\mathbf{R}^+ \times \widehat{\mathbf{Z}}^*)$ by the pullback

$$(2.1) \quad \psi^* : \mathbf{A}^* \longrightarrow \widehat{\mathbf{Z}}^* \longrightarrow (\mathbf{Z}/N\mathbf{Z})^* \longrightarrow \mathbf{C}^*,$$

where the first arrows are the canonical projections, and the last arrow is ψ . We drop the $*$ from the notation for the local constituents. Thus $\psi_p : \mathbf{Q}_p^* \rightarrow \mathbf{C}^*$ is given by restricting ψ^* to the embedded image of \mathbf{Q}_p^* in \mathbf{A}^* . Note that if d is an integer prime to N , then

$$(2.2) \quad \psi(d) = \prod_{p|N} \psi_p(d) = \psi^*(d_N).$$

Later we will consider a character χ of modulus D , and all of the above notation will apply equally with D in place of N .

We let $\theta : \mathbf{A} \rightarrow \mathbf{C}^*$ denote the standard character of \mathbf{A} , given locally by

$$\theta_p(x) = \begin{cases} e^{-2\pi i x} & \text{if } p = \infty \quad (x \in \mathbf{R}) \\ e^{2\pi i r_p(x)} & \text{if } p < \infty \quad (x \in \mathbf{Q}_p), \end{cases}$$

where $r_p(x) \in \mathbf{Q}$ is the p -principal part of x , a number with p -power denominator characterized up to \mathbf{Z} by $x \in r_p(x) + \mathbf{Z}_p$. The global character $\theta = \prod_{p \leq \infty} \theta_p$ is then

trivial on \mathbf{Q} , and for finite p , θ_p is trivial precisely on \mathbf{Z}_p . For $r \in \mathbf{Q}$, we define

$$(2.3) \quad \theta_r(x) = \theta(-rx) = \overline{\theta(rx)}.$$

Every character of $\mathbf{Q} \backslash \mathbf{A}$ is of the form θ_r for some $r \in \mathbf{Q}$.

Let G denote the algebraic group GL_2 , with center Z , and let \overline{G} denote G/Z . The group $G(\mathbf{A}_{\mathrm{fin}})$ has the following sequences of open compact subgroups of $K = G(\widehat{\mathbf{Z}})$:

$$K_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K \mid c \in N\widehat{\mathbf{Z}} \right\}$$

$$K_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(N) \mid d \equiv 1 \pmod{N\widehat{\mathbf{Z}}} \right\}.$$

Because $\det K_0(N) = \det K_1(N) = \widehat{\mathbf{Z}}^*$, strong approximation holds for both of these, and in particular,

$$(2.4) \quad G(\mathbf{A}) = G(\mathbf{Q})(G(\mathbf{R})^+ \times K_1(N)).$$

Let $L^2(\psi^*) = L^2(\overline{G}(\mathbf{Q}) \backslash \overline{G}(\mathbf{A}), \psi^*)$ be the space of measurable \mathbf{C} -valued functions ϕ on $G(\mathbf{A})$ satisfying $\phi(z\gamma g) = \psi^*(z)\phi(g)$ for all $z \in Z(\mathbf{A})$, $\gamma \in G(\mathbf{Q})$, $g \in G(\mathbf{A})$, and which are square integrable over $\overline{G}(\mathbf{Q}) \backslash \overline{G}(\mathbf{A})$. Let $L_0^2(\psi^*)$ denote the subspace of cuspidal functions.

We now normalize Haar measure on each group of interest. Everything is the same as in §7 of [KL1], where more detail is given. On \mathbf{R} we take Lebesgue measure dx , and we take $\frac{dy}{|y|}$ on \mathbf{R}^* . We normalize the additive measure dx on \mathbf{Q}_p by taking $\mathrm{meas}(\mathbf{Z}_p) = 1$, and likewise d^*y on \mathbf{Q}_p^* is normalized by $\mathrm{meas}(\mathbf{Z}_p^*) = 1$. These choices determine Haar measures on \mathbf{A} and \mathbf{A}^* in the usual way, with the property that $\mathrm{meas}(\mathbf{A}/\mathbf{Q}) = 1$. We give the compact abelian group $K_\infty = \mathrm{SO}(2)$ the measure dk of total length 1, and use the above measures to define measures on $N(\mathbf{R}) = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\} \cong \mathbf{R}$ and $M(\mathbf{R}) = \left\{ \begin{pmatrix} y & \\ & z \end{pmatrix} \right\} \cong \mathbf{R}^* \times \mathbf{R}^*$. These choices determine a Haar measure on $G(\mathbf{R})$ by the Iwasawa decomposition: $dg = d(mnk) = dm \, dn \, dk$. In the same way, our fixed measures on \mathbf{Q}_p , \mathbf{Q}_p^* determine measures on $N(\mathbf{Q}_p)$ and $M(\mathbf{Q}_p)$ respectively. We take the unique measure on $G(\mathbf{Q}_p)$ for which the open compact subgroup $K_p = G(\mathbf{Z}_p)$ has measure 1. Let Z denote the center of G and set $\overline{G} = G/Z$. We take $\mathrm{meas}(\overline{K}_p) = 1$ in $\overline{G}(\mathbf{Q}_p)$. On $\overline{G}(\mathbf{R})$ we take the measure $d\overline{m} \, dn \, dk$, where $d\overline{m}$ is the measure $\frac{dy}{|y|}$ on $\overline{M}(\mathbf{R}) \cong \left\{ \begin{pmatrix} y & \\ & 1 \end{pmatrix} \right\} \cong \mathbf{R}^*$. These local measures determine a Haar measure on $\overline{G}(\mathbf{A})$ for which $\mathrm{meas}(\overline{G}(\mathbf{Q}) \backslash \overline{G}(\mathbf{A})) = \pi/3$.

Having fixed the measure, we note the following.

Lemma 2.1. *Let $D > 0$ and let χ be a Dirichlet character modulo D (not necessarily primitive), with Gauss sum $\tau(\chi)$ as in (1.3). Let χ^* be the adelic realization of χ as in (2.1). Then for any integer n prime to D ,*

$$(2.5) \quad \int_{\widehat{\mathbf{Z}}^*} \chi^*(u) \theta_{\mathrm{fin}}\left(\frac{nu}{D}\right) d^*u = \frac{\overline{\chi(n)}}{\varphi(D)} \tau(\chi),$$

where φ is Euler's φ -function and $\theta_{\mathrm{fin}} = \prod_{p < \infty} \theta_p$.

Proof. The integrand in (2.5) is invariant under the subgroup

$$U_D = (1 + D\widehat{\mathbf{Z}}) \cap \widehat{\mathbf{Z}}^* = \prod_{p|D} (1 + D\mathbf{Z}_p) \prod_{p \nmid D} \mathbf{Z}_p^*.$$

Note that $\widehat{\mathbf{Z}}^*/U_D \cong (\mathbf{Z}/D\mathbf{Z})^*$, so $\text{meas}(U_D) = \varphi(D)^{-1}$. Therefore

$$\begin{aligned} \int_{\widehat{\mathbf{Z}}^*} \chi^*(u) \theta_{\text{fin}}\left(\frac{nu}{D}\right) d^*u &= \frac{1}{\varphi(D)} \sum_{m \in (\mathbf{Z}/D\mathbf{Z})^*} \chi^*(m_D) \theta_{\text{fin}}\left(\frac{nm}{D}\right) \\ &= \frac{1}{\varphi(D)} \sum_{m \bmod D} \chi(m) e^{2\pi i nm/D} = \overline{\chi(n)} \frac{\tau(\chi)}{\varphi(D)}. \end{aligned} \quad \square$$

We let

$$G(\mathbf{R})^+ = \{g \in G(\mathbf{R}) \mid \det g > 0\}.$$

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathbf{R})^+$, we set

$$j(g, z) = \det(g)^{-1/2} (cz + d).$$

Recall that $j(g_1 g_2, z) = j(g_1, g_2 z) j(g_2, z)$. The group action of $G(\mathbf{R})^+$ on the complex upper half-plane \mathbf{H} by linear fractional transformations extends to a right action on the space of functions $h : \mathbf{H} \rightarrow \mathbf{C}$ via the weight k slash operator

$$h|_g(z) = j(g, z)^{-k} h(g(z)) \quad (g \in G(\mathbf{R})^+, z \in \mathbf{H}).$$

Fix a Dirichlet character ψ of modulus N , a positive integer k satisfying

$$(2.6) \quad \psi(-1) = (-1)^k,$$

and let $S_k(N, \psi)$ denote the space of cusp forms of level N , weight k , and character ψ . Thus $h \in S_k(N, \psi)$ satisfies

$$(2.7) \quad h\left|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right. = \psi(d) h$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ and $z \in \mathbf{H}$.

The adelization of h is the function $\phi_h \in L_0^2(\overline{\psi^*})$ defined using strong approximation (2.4) by

$$(2.8) \quad \phi_h(\gamma(g_\infty \times g_{\text{fin}})) = j(g_\infty, i)^{-k} h(g_\infty(i))$$

for $\gamma \in G(\mathbf{Q})$, $g_\infty \in G(\mathbf{R})^+$, and $g_{\text{fin}} \in K_1(N)$. The modularity of h makes ϕ_h well-defined, and one checks readily that the central character is indeed $\overline{\psi^*}$ (see e.g. the proof of Proposition 4.5 of [KL3]; the complex conjugate is needed here because we have not included it in (2.7).) With the choice of Haar measure on $\overline{G}(\mathbf{A})$ given above, and the normalization (1.1), the map $h \mapsto \phi_h$ is an isometry, i.e. $\|h\| = \|\phi_h\|$ (cf. (12.20) of [KL1]).

We recall the meaning of the following “period integrals”.

Lemma 2.2. *For $r \in \mathbf{Q}$, and $h \in S_k(N, \psi)$,*

$$\int_{\mathbf{Q} \backslash \mathbf{A}} \phi_h\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) \overline{\theta_r(x)} dx = \begin{cases} e^{-2\pi r} a_r(h) & \text{if } r \in \mathbf{Z}^+, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\int_{\mathbf{Q}^* \backslash \mathbf{A}^*} \phi_h\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) |y|^{s-k/2} d^*y = \Lambda(s, h).$$

Proof. See e.g. [KL1] Corollary 12.4 and [KL2] Lemma 3.1 respectively. \square

3. THE TWISTING OPERATOR

From now on, we assume that χ is a primitive Dirichlet character modulo D , with $(D, nNr) = 1$. Recall that $L(s, h, \chi) = L(s, h_\chi)$, where $h_\chi \in S_k(D^2N, \chi^2\psi)$ is given by

$$h_\chi(z) = \sum_{n=1}^{\infty} \chi(n) a_n(h) e^{2\pi i n z},$$

or equivalently,

$$(3.1) \quad h_\chi = \frac{1}{\tau(\overline{\chi})} \sum_{m \bmod D} \overline{\chi(m)} h|_{\begin{pmatrix} 1 & m/D \\ 0 & 1 \end{pmatrix}}$$

(see e.g. [Bu], p. 59). Likewise, it follows from the definitions that for $g_\infty \in G(\mathbf{R})^+$,

$$(3.2) \quad \phi_{h_\chi}(g_\infty \times 1_{\text{fin}}) = \frac{1}{\tau(\overline{\chi})} \sum_{m \bmod D} \overline{\chi(m)} \phi_h\left(\begin{pmatrix} 1 & m/D \\ 0 & 1 \end{pmatrix} g_\infty \times 1_{\text{fin}}\right).$$

Because χ is assumed to be primitive, we have $|\tau(\overline{\chi})| = \sqrt{D}$.

We now define a test function $f^\chi : G(\mathbf{A}_{\text{fin}}) \rightarrow \mathbf{C}$ which essentially realizes the twisting map $h \mapsto h_\chi$ adelically (see also [RR]). It will be supported on the disjoint union

$$(3.3) \quad \text{Supp}(f^\chi) = \bigcup_{\substack{m \bmod D, \\ (m, D)=1}} \begin{pmatrix} 1 & -m/D \\ 0 & 1 \end{pmatrix} Z(\mathbf{A}_{\text{fin}}) K_1(N),$$

where the rational matrix is embedded diagonally in $G(\mathbf{A}_{\text{fin}})$. The value of f^χ on the coset indexed by m is defined to be

$$(3.4) \quad f^\chi\left(\begin{pmatrix} 1 & -m/D \\ 0 & 1 \end{pmatrix} zk\right) = \frac{\nu(N) \overline{\chi(m)} \psi^*(z)}{\tau(\overline{\chi})}.$$

Here as before,

$$(3.5) \quad \nu(N) = [K : K_0(N)] = [\overline{K} : \overline{K_1(N)}] = \text{meas}(\overline{K_1(N)})^{-1}.$$

Lemma 3.1. *Consider the open compact subgroup*

$$J = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_1(D^2N) \mid a \equiv 1 \pmod{D\widehat{\mathbf{Z}}} \right\}.$$

The function f^χ is right $K_1(N)$ -invariant and left J -invariant.

Proof. The first claim is obvious from the definition of f^χ . The second claim follows from the fact that

$$\begin{pmatrix} a & b \\ cD^2N & d \end{pmatrix} \begin{pmatrix} 1 & -\frac{m}{D} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{m}{D} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a + mcD^2N & b + (d-a)\frac{m}{D} - m^2cN \\ cD^2N & d - mcD^2N \end{pmatrix},$$

noting that if $a \equiv 1 \pmod{D}$ and $d \equiv 1 \pmod{D^2N}$, the matrix on the right belongs to $K_1(N)$. \square

Note that because f^χ has compact support modulo the center, and transforms under $Z(\mathbf{A}_{\text{fin}})$ by ψ^* , it defines an operator on $L^2(\overline{\psi^*})$ by

$$(3.6) \quad R(f^\chi)\phi(x) = \int_{G(\mathbf{A}_{\text{fin}})} f^\chi(g)\phi(xg)dg.$$

This operator is closely related to the twisting function $h \mapsto h_\chi$, as we now show.

Proposition 3.2. *Let $h \in S_k(N, \psi)$ and let χ^* be the Hecke character attached to χ as in (2.1). Then for all $x \in G(\mathbf{A}_{\text{fin}})$,*

$$(3.7) \quad R(f^\chi)\phi_h(x) = \chi^*(a_D)\phi_{h_\chi}(x),$$

where a is determined from x using strong approximation by writing

$$x = x_{\mathbf{Q}}(x_\infty \times \begin{pmatrix} a & b \\ c & d \end{pmatrix})$$

for $x_{\mathbf{Q}} \in G(\mathbf{Q})$, $x_\infty \in G(\mathbf{R})^+$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_1(D^2N)$, and a_D is the finite idele with local components $(a_D)_p = a_p$ (resp. 1) if $p|D$ (resp. $p \nmid D$).

Remark: If $a \in \widehat{\mathbf{Z}}^*$, then $\chi^*(a_D) = \chi^*(a)$.

Proof. Clearly $R(f^\chi)\phi_h$ inherits the left $G(\mathbf{Q})$ -invariance from ϕ_h . Hence we can assume that $x_{\mathbf{Q}} = 1$. It is an easy consequence of the above lemma that $R(f^\chi)\phi$ is right J -invariant. Therefore we may modify x_{fin} on the right by an appropriate element of J to reduce to the case where $x_{\text{fin}} = \begin{pmatrix} a_D & \\ & 1 \end{pmatrix}$. Hence it suffices to prove that

$$R(f^\chi)\phi_h(x_\infty \times \begin{pmatrix} a_D & \\ & 1 \end{pmatrix}) = \chi^*(a_D)\phi_{h_\chi}(x_\infty \times 1_{\text{fin}}).$$

Let $\alpha_m = \begin{pmatrix} 1 & m/D \\ 0 & 1 \end{pmatrix}$. From the preceding definitions and the right $K_1(N)$ -invariance of ϕ_h , for any $x \in G(\mathbf{A})$ we have

$$\begin{aligned} R(f^\chi)\phi_h(x) &= \sum_{m \bmod D} \int_{\alpha_m^{-1}K_1(N)} f^\chi(g)\phi_h(xg)dg \\ &= \sum_{m \bmod D} \phi_h(x\alpha_m^{-1}) \frac{\nu(N)\overline{\chi(m)}}{\tau(\overline{\chi})} \int_{\alpha_m^{-1}K_1(N)} dg \\ &= \frac{1}{\tau(\overline{\chi})} \sum_{m \bmod D} \overline{\chi(m)}\phi_h(x\alpha_m^{-1}). \end{aligned}$$

Taking $x = x_\infty \times \begin{pmatrix} a_D & \\ & 1 \end{pmatrix}$ we have

$$\begin{aligned} \phi_h(x\alpha_m^{-1}) &= \phi_h(x_\infty \times \begin{pmatrix} a_D & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{m}{D} \\ 0 & 1 \end{pmatrix}) = \phi_h(x_\infty \times \begin{pmatrix} 1 & -\frac{a_D m}{D} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_D & \\ & 1 \end{pmatrix}) \\ &= \phi_h(x_\infty \times \begin{pmatrix} 1 & -\frac{a_D m}{D} \\ 0 & 1 \end{pmatrix}) \end{aligned}$$

by the right $K_1(N)$ -invariance of ϕ_h . Fix an integer z relatively prime to D such that $z \equiv a_D \pmod{D\widehat{\mathbf{Z}}}$, and multiply through on the left by α_{zm} . By the left $G(\mathbf{Q})$ -invariance this has no effect, so the above is

$$= \phi_h(\begin{pmatrix} 1 & \frac{zm}{D} \\ 0 & 1 \end{pmatrix} x_\infty \times \begin{pmatrix} 1 & \frac{(z-a_D)m}{D} \\ 0 & 1 \end{pmatrix}) = \phi_h(\alpha_{zm}x_\infty \times 1_{\text{fin}}),$$

again by right $K_1(N)$ -invariance. Therefore

$$\begin{aligned} R(f^\chi)\phi_h(x_\infty \times \begin{pmatrix} a_D & \\ & 1 \end{pmatrix}) &= \frac{1}{\tau(\overline{\chi})} \sum_{m \bmod D} \overline{\chi(m)}\phi_h(\alpha_{zm}x_\infty \times 1_{\text{fin}}) \\ &= \frac{\chi(z)}{\tau(\overline{\chi})} \sum_{m \bmod D} \overline{\chi(m)}\phi_h(\alpha_m x_\infty \times 1_{\text{fin}}) = \chi(z)\phi_{h_\chi}(x_\infty \times 1_{\text{fin}}) \end{aligned}$$

by (3.2). This gives the desired result, since $\chi(z) = \chi^*(a_D)$ by (2.1) and (2.2). \square

It will be useful to define local components for f^χ . For any prime p , we have defined χ_p to be the character of \mathbf{Q}_p^* attached to the Hecke character χ^* . When $p|D$, we have

$$\chi_p : \mathbf{Z}_p^* \longrightarrow (\mathbf{Z}_p/D\mathbf{Z}_p)^* \cong (\mathbf{Z}/p^{D_p}\mathbf{Z})^* \longrightarrow \mathbf{C}^*.$$

Still assuming $p|D$, a local version of (2.5) is the following:

$$(3.8) \quad \int_{\mathbf{Z}_p^*} \chi_p(u) \theta_p\left(\frac{nu}{D}\right) d^*u = \frac{\overline{\chi_p(n)}}{\varphi(p^{D_p})} \tau(\chi)_p,$$

where

$$(3.9) \quad \tau(\chi)_p = \chi_p\left(\frac{D}{p^{D_p}}\right) \tau(\chi_p)$$

for $\tau(\chi_p) = \sum_{m \in (\mathbf{Z}/p^{D_p}\mathbf{Z})^*} \chi_p(m) e^{2\pi i m/p^{D_p}}$ the Gauss sum of the character χ_p . Then

by (3.8) and (2.5), we have

$$(3.10) \quad \tau(\chi) = \prod_{p|D} \tau(\chi)_p.$$

Given a prime p , we define a local function $f_p^\chi : G(\mathbf{Q}_p) \longrightarrow \mathbf{C}$ as follows. If $p|D$, we take

$$(3.11) \quad \text{Supp}(f_p^\chi) = \bigcup_{\substack{m \bmod D\mathbf{Z}_p \\ p \nmid m}} \begin{pmatrix} 1 & -m/D \\ 0 & 1 \end{pmatrix} Z_p K_p$$

(a disjoint union), and define $f_p^\chi = \sum_m f_{p,m}^\chi$, where $f_{p,m}^\chi$ is supported on the coset indexed by m in (3.11), and is given by

$$(3.12) \quad f_{p,m}^\chi\left(\begin{pmatrix} 1 & -m/D \\ 0 & 1 \end{pmatrix} zk\right) = \frac{\overline{\chi_p(m)} \psi_p(z)}{\tau(\overline{\chi})_p}$$

for $\tau(\overline{\chi})_p$ as in (3.9). The value is independent of the choice of representative for $m \in (\mathbf{Z}_p/p^{D_p}\mathbf{Z}_p)^*$ since χ_p has conductor p^{D_p} . If $p \nmid D$, then f_p^χ is supported on $Z_p K_1(N)_p$, and we define it by

$$f_p^\chi(zk) = \nu_p(N) \psi_p(z),$$

where $\nu_p(N) = [K_p : K_0(N)_p] = \nu(p^{N_p})$. It is easily verified using (2.2) (applied to χ^*) and (3.10) that $f^\chi = \prod_p f_p^\chi$.

4. THE HECKE OPERATOR

We refer to §13 of [KL1] for a more detailed account of the adelic Hecke operator defined here. Fix a positive integer n with $(n, DN) = 1$. Define

$$M(n, N) = \{g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\widehat{\mathbf{Z}}) \mid \det g \in n\widehat{\mathbf{Z}}^*, c \in N\widehat{\mathbf{Z}}\}.$$

Define a function $f^n : G(\mathbf{A}_{\text{fin}}) \longrightarrow \mathbf{C}$ with

$$\text{Supp}(f^n) = Z(\mathbf{A}_{\text{fin}})M(n, N) = Z(\mathbf{Q}^+)M(n, N)$$

by

$$f^n(z_{\mathbf{Q}}m) = \nu(N) \psi^*(d_N) \quad (z_{\mathbf{Q}} \in Z(\mathbf{Q}^+), m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(n, N)).$$

One shows easily that f^n is bi- $K_1(N)$ -invariant.

For any prime p , let

$$M(n, N)_p = \{g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{Z}_p) \mid \det g \in n\mathbf{Z}_p^*, c \in N\mathbf{Z}_p\}.$$

Notice that if $p \nmid n$ then $M(n, N)_p = K_0(N)_p$. Define a function $f_p^n : G(\mathbf{Q}_p) \rightarrow \mathbf{C}$, supported on $Z(\mathbf{Q}_p)M(n, N)_p$, by

$$(4.1) \quad f_p^n(zm) = \nu_p(N)\psi_p(z)\psi_p((d_N)_p) \quad (z \in Z(\mathbf{Q}_p), m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(n, N)_p).$$

Then it is straightforward to check that $f^n(g) = \prod_p f_p^n(g_p)$ for $g \in G(\mathbf{A}_{\text{fin}})$.

Proposition 4.1. *For $h \in S_k(N, \psi)$,*

$$(4.2) \quad R(f^n)\phi_h = n^{1-k/2}\phi_{T_n h}.$$

Proof. See [KL1], Proposition 13.6. □

5. THE GLOBAL TEST FUNCTION

We take $f_\infty(g) = d_k \overline{\langle \pi_k(g)v_0, v_0 \rangle}$, where π_k is the weight k discrete series representation of $\text{GL}_2(\mathbf{R})$ with formal degree $d_k = \frac{k-1}{4\pi}$, central character $\begin{pmatrix} x & \\ & x \end{pmatrix} \mapsto \text{sgn}(x)^k$, and lowest weight unit vector v_0 . Explicitly,

$$f_\infty\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \frac{(k-1)}{4\pi} \frac{\det(g)^{k/2}(2i)^k}{(-b+c+(a+d)i)^k}$$

if $ad - bc > 0$, and it vanishes otherwise (see [KL1], Theorem 14.5). This function is self-adjoint, meaning

$$(5.1) \quad f_\infty(g) = \overline{f_\infty(g^{-1})}.$$

It is integrable if and only if $k > 2$ ([KL1], Prop. 14.3).

Given two functions $f_1, f_2 \in L^1(\psi^*)$, we define their convolution

$$f_1 * f_2(x) = \int_{\overline{G}(\mathbf{A})} f_1(g)f_2(g^{-1}x)dg = \int_{\overline{G}(\mathbf{A})} f_1(xg^{-1})f_2(g)dg.$$

It is straightforward to show that

$$(5.2) \quad R(f_1 * f_2) = R(f_1) \circ R(f_2)$$

as operators on $L^2(\overline{\psi^*})$, where

$$R(f)\phi(x) = \int_{\overline{G}(\mathbf{A})} f(g)\phi(xg)dg.$$

Fix an integer $n > 0$ relatively prime to DN , and set

$$(5.3) \quad f = (f_\infty \times f^\chi) * (f_\infty \times f^n).$$

Local components for f can be defined as follows.

Proposition 5.1. *With notation as in the previous two sections, define*

$$f_p = \begin{cases} f_p^\chi = f_p^n & \text{if } p \nmid nD \\ f_p^\chi & \text{if } p \mid D \\ f_p^n & \text{if } p \mid n. \end{cases}$$

Then $f = f_\infty \prod_p f_p$.

Proof. Because $f_\infty \times f^\chi$ and $f_\infty \times f^n$ are both factorizable and identically 1 on K_p for a.e. p , the integral defining their convolution is factorizable, and hence

$$f = (f_\infty \times f^\chi) * (f_\infty \times f^n) = (f_\infty * f_\infty) \prod_p (f_p^\chi * f_p^n).$$

It follows directly from the orthogonality relations for discrete series that $f_\infty * f_\infty = f_\infty$. Indeed,

$$\begin{aligned} f_\infty * f_\infty(x) &= d_k^2 \int_{\overline{G}(\mathbf{R})} \overline{\langle \pi_k(g)v_0, v_0 \rangle} \langle \pi_k(g^{-1}x)v_0, v_0 \rangle dg \\ &= d_k^2 \int_{\overline{G}(\mathbf{R})} \langle \pi_k(g)v_0, \pi_k(x)v_0 \rangle \overline{\langle \pi_k(g)v_0, v_0 \rangle} dg = d_k^2 \frac{\langle v_0, v_0 \rangle \overline{\langle \pi_k(x)v_0, v_0 \rangle}}{d_k} = f_\infty(x). \end{aligned}$$

Likewise, simple direct computation shows that for finite primes p , $f_p^\chi * f_p^n = f_p$ as given. \square

Globally, the support of f is

$$(5.4) \quad \text{Supp}(f) = G(\mathbf{R})^+ \times \bigcup_{\substack{m \bmod D \\ (m,D)=1}} \begin{pmatrix} 1 & -m/D \\ 0 & 1 \end{pmatrix} Z(\mathbf{Q}^+) M(n, N).$$

The union over m is easily seen to be disjoint, using the fact that $(n, D) = 1$. Accordingly, we can write

$$f_{\text{fin}} = \sum_{m \in (\mathbf{Z}/D\mathbf{Z})^*} f_m,$$

where f_m is supported on the coset indexed by m in (5.4), and

$$(5.5) \quad f_m\left(\begin{pmatrix} 1 & -m/D \\ 0 & 1 \end{pmatrix} zk\right) = \frac{\nu(N) \chi(\overline{m}) \psi^*(d_N)}{\tau(\overline{\chi})}$$

for $z \in Z(\mathbf{Q}^+)$ and $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(n, N)$.

Under the condition $k \geq 2$ (which will be in force throughout) $f \in L^1(\psi^*)$, so the operator $R(f)$ on $L^2(\overline{\psi^*})$ is defined.

Proposition 5.2. *The operator $R(f)$ factors through the orthogonal projection of $L^2(\overline{\psi^*})$ onto the finite dimensional subspace $S_k(N, \psi)$ (embedded in $L^2(\overline{\psi^*})$ via (2.8)).*

Proof. The operator $R(f_\infty \times f^n)$ factors through the orthogonal projection onto $S_k(N, \psi)$ ([KL1], Corollary 13.13). Therefore by (5.2) and (5.3), $R(f)$ has the same property. \square

Remark: The image of $R(f)$ is *not* contained in any classical space of cusp forms $S_k(\Gamma_1(M))$. Indeed, it is immediate from (3.7) that the image of $R(f)$ is not left $K_1(M)$ -invariant for any M , since a congruence condition on a is needed.

The operator $R(f)$ is an integral operator given by the continuous kernel

$$(5.6) \quad K(g_1, g_2) = \sum_{h \in \mathcal{F}} \frac{R(f) \phi_h(g_1) \overline{\phi_h(g_2)}}{\|h\|^2} = \sum_{\gamma \in \overline{G}(\mathbf{Q})} f(g_1^{-1} \gamma g_2).$$

Here the spectral sum is taken over any orthogonal basis \mathcal{F} for $S_k(N, \psi)$, as a consequence of Proposition 5.2, and both sums are absolutely convergent.

6. SPECTRAL SIDE

We prove the theorem by computing the following integral

$$(6.1) \quad \int_{\mathbf{Q}^* \backslash \mathbf{A}^*} \int_{\mathbf{Q} \backslash \mathbf{A}} K\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \theta_r(x) \overline{\chi^*(y)} |y|^{s-k/2} dx d^*y$$

using the two expressions for the kernel (5.6). We note that the double integral is absolutely convergent for all $s \in \mathbf{C}$ (see below).

On the spectral side, (6.1) becomes

$$(6.2) \quad \sum_{h \in \mathcal{F}} \frac{1}{\|h\|^2} \int_{\mathbf{Q}^* \backslash \mathbf{A}^*} R(f) \phi_h\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) \overline{\chi^*(y)} |y|^{s-k/2} d^*y \cdot \int_{\mathbf{Q} \backslash \mathbf{A}} \overline{\phi_h\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)} \theta_r(x) dx.$$

Choose \mathcal{F} in (5.6) to consist of eigenvectors of T_n . Then for $h \in \mathcal{F}$, we write $T_n h = \lambda_n(h) h$, so that by (3.7) and (4.2),

$$R(f) \phi_h\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) = n^{1-k/2} \lambda_n(h) \chi^*(y) \phi_{h_\chi}\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right)$$

for all $y \in \mathbf{R}^+ \times \widehat{\mathbf{Z}}^* \cong \mathbf{Q}^* \backslash \mathbf{A}^*$. Consequently the factor $\overline{\chi^*(y)}$ in (6.2) is cancelled out, and (6.2) becomes

$$(6.3) \quad \sum_{h \in \mathcal{F}} \frac{n^{1-k/2} \lambda_n(h)}{\|h\|^2} \int_{\mathbf{Q}^* \backslash \mathbf{A}^*} \phi_{h_\chi}\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) |y|^{s-k/2} d^*y \int_{\mathbf{Q} \backslash \mathbf{A}} \overline{\phi_h\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)} \theta_r(x) dx$$

$$(6.4) \quad = \frac{n^{1-k/2}}{e^{2\pi r}} \sum_{h \in \mathcal{F}} \frac{\lambda_n(h) \overline{a_r(h)}}{\|h\|^2} \Lambda(s, h, \chi)$$

by Lemma 2.2. The two integrals in (6.3) are absolutely convergent for all s , so we have the following.

Proposition 6.1. *The double integral (6.1) is absolutely convergent for all $s \in \mathbf{C}$.*

7. GEOMETRIC SIDE

For the moment, let $H(\mathbf{A}) = \overline{M}(\mathbf{A}) \times N(\mathbf{A}) \cong \mathbf{A}^* \times \mathbf{A}$. Inserting the geometric expression $K(g_1, g_2) = \sum_{\gamma} f(g_1^{-1} \gamma g_2)$ into (6.1), we get

$$\begin{aligned} & \int_{H(\mathbf{Q}) \backslash H(\mathbf{A})} \sum_{\gamma \in \overline{G}(\mathbf{Q})} f\left(\begin{pmatrix} y^{-1} & 0 \\ 0 & 1 \end{pmatrix} \gamma \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \theta_r(x) \overline{\chi^*(y)} |y|^{s-k/2} dx d^*y \\ &= \int_{H(\mathbf{Q}) \backslash H(\mathbf{A})} \sum_{\delta} \sum_{\gamma \in [\delta]} f\left(\begin{pmatrix} y^{-1} & 0 \\ 0 & 1 \end{pmatrix} \gamma \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \theta_r(x) \overline{\chi^*(y)} |y|^{s-k/2} dx d^*y, \end{aligned}$$

where δ ranges over a set of representatives for the $H(\mathbf{Q})$ -orbits in $\overline{G}(\mathbf{Q})$ relative to the action $(m, n) \cdot \gamma = m^{-1} \gamma n$, and $[\delta] = \{m^{-1} \delta n \mid (m, n) \in H_\delta(\mathbf{Q}) \backslash H(\mathbf{Q})\}$ is the orbit. It is not hard to check that in fact for all δ , the stabilizer $H_\delta(\mathbf{Q}) = \{1\}$. Therefore, (6.1) is formally equal to

$$(7.1) \quad \sum_{\delta} \int_{\mathbf{A}^*} \int_{\mathbf{A}} f\left(\begin{pmatrix} y^{-1} & \\ & 1 \end{pmatrix} \delta \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) \theta_r(x) \overline{\chi^*(y)} |y|^{s-k/2} dx d^*y,$$

where δ runs through a set of representatives for $\overline{M}(\mathbf{Q}) \backslash \overline{G}(\mathbf{Q}) / N(\mathbf{Q})$. By the Bruhat decomposition

$$G(\mathbf{Q}) = M(\mathbf{Q}) N(\mathbf{Q}) \cup M(\mathbf{Q}) N(\mathbf{Q}) \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} N(\mathbf{Q}),$$

a set of representatives δ is given by

$$(7.2) \quad \{1\} \cup \left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} t & -1 \\ 1 & 0 \end{pmatrix} \mid t \in \mathbf{Q}^* \right\}.$$

The equality between (6.1) and (7.1) is valid on the strip $1 < \operatorname{Re}(s) < k-1$. This is a consequence of the following.

Proposition 7.1. *Suppose $1 < \operatorname{Re}(s) < k-1$. Then*

$$\sum_{\delta} \int_{\mathbf{A}^*} \int_{\mathbf{A}} \left| f\left(\begin{pmatrix} y^{-1} & \\ & 1 \end{pmatrix} \delta \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) \theta_r(x) \overline{\chi^*(y)} |y|^{s-k/2} \right| dx d^*y < \infty.$$

Proof. This is proven in just the same way as the analogous result in [KL2], Proposition 3.3. We outline the steps. Because f_{fin} is bounded and compactly supported as a function of y, x , the argument hinges on bounding the infinite part

$$I_{\delta}^{abs}(f)_{\infty} = \int_0^{\infty} \int_{-\infty}^{\infty} |f_{\infty}\left(\begin{pmatrix} y^{-1} & \\ & 1 \end{pmatrix} \delta \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right)| dx y^{\sigma-k/2-1} dy,$$

where $\sigma = \operatorname{Re}(s)$. However, by (5.1) the above is

$$= \int_0^{\infty} \int_{-\infty}^{\infty} |f_{\infty}\left(\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \delta^{-1} \begin{pmatrix} y & \\ & 1 \end{pmatrix}\right)| dx y^{\sigma-k/2-1} dy.$$

Noting that the set of δ in (7.2) is exactly the set of inverses of the δ 's in [KL2], the above coincides with $I_{\delta^{-1}}^{abs}(f)_{\infty}$ considered in §3.3 there. Hence those results give

$$I_{\delta}^{abs}(f) < \infty \quad \text{for} \quad \begin{cases} \delta = 1 \text{ and } 0 < \sigma < k-1 \\ \delta = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \text{ and } 1 < \sigma < k \\ \delta = \begin{pmatrix} t & -1 \\ 1 & 0 \end{pmatrix} \text{ and } 0 < \sigma < k. \end{cases}$$

Furthermore, by Proposition 3.3 of [KL2], for $\delta_t = \begin{pmatrix} t & -1 \\ 1 & 0 \end{pmatrix}$ we have

$$(7.3) \quad I_{\delta_t}^{abs}(f)_{\infty} \ll |t|^{\sigma-k} \quad \text{if } 0 < \sigma < k.$$

Thus, to complete the proof it remains to show that for $1 < \sigma < k-1$,

$$\sum_{t \in \mathbf{Q}^*} I_{\delta_t}^{abs}(f) < \infty.$$

We will prove in Proposition 8.1 below that the finite part $I_{\delta_t}^{abs}(f)_{\text{fin}}$ vanishes unless $t = \frac{Nb}{nD}$ for some $b \in \mathbf{Z} - \{0\}$. We will show in (8.14) that for such t ,

$$|I_{\delta_t}^{abs}(f)_{\text{fin}}| \leq \frac{n^{\sigma-k/2} \nu(N) \varphi(D) \gcd(r, n)}{N^{2\sigma-k} D^{1/2}} \sum_{d|b} d^{k-2\sigma}.$$

Together with (7.3), the fact that $\#\{d : d|b\} \ll |b|^{\varepsilon}$ for $\varepsilon > 0$, and using $d^{k-2\sigma} \leq 1$ when $\sigma > k/2$, this gives the global estimate

$$\sum_{t \in \mathbf{Q}^*} I_{\delta_t}^{abs}(f) \ll_{N,D,n,\varepsilon} \begin{cases} \sum_{b \in \mathbf{Z} - \{0\}} |b|^{-\sigma+\varepsilon} & \text{if } \sigma \leq k/2 \\ \sum_{b \in \mathbf{Z} - \{0\}} |b|^{\sigma-k+\varepsilon} & \text{if } \sigma > k/2. \end{cases}$$

This is finite when $1 < \sigma < k-1$ and ε is sufficiently small. \square

Let $I_{\delta}(s)$ denote the double integral attached to δ in (7.1). For $1 < \operatorname{Re}(s) < k-1$ and each δ in (7.2), we need to compute $I_{\delta}(s)$. It factorizes as

$$I_{\delta}(s) = I_{\delta}(s)_{\infty} I_{\delta}(s)_{\text{fin}} = I_{\delta}(s)_{\infty} \prod_p I_{\delta}(s)_p,$$

where

$$I_\delta(s)_\infty = \int_{\mathbf{R}^*} \int_{\mathbf{R}} f_\infty\left(\begin{pmatrix} y^{-1} & \\ & 1 \end{pmatrix} \delta\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \overline{\theta_\infty(rx)\chi_\infty(y)} |y|^{s-k/2} dx \frac{dy}{|y|}$$

and likewise

$$I_\delta(s)_p = \int_{\mathbf{Q}_p^*} \int_{\mathbf{Q}_p} f_p\left(\begin{pmatrix} y^{-1} & \\ & 1 \end{pmatrix} \delta\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \overline{\theta_p(rx)\chi_p(y)} |y|_p^{s-k/2} dx d^*y.$$

From the definition of f_∞ , the integrand for $I_\delta(s)_\infty$ vanishes unless $y > 0$. Because χ_∞ is trivial on \mathbf{R}^+ , it has no effect on $I_\delta(s)_\infty$ and can be removed. Using (5.1), we find that

$$(7.4) \quad I_\delta(s)_\infty = \overline{I'_{\delta-1}(\overline{s})_\infty},$$

where

$$I'_{\delta-1}(s)_\infty = \int_0^\infty \int_{-\infty}^\infty f_\infty\left(\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \delta^{-1}\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) \theta_\infty(rx) y^{s-k/2} dx \frac{dy}{y}$$

is the archimedean factor computed in [KL2].

For convenience, when computing the finite part $I_\delta(s)_{\text{fin}}$ we will replace y by y^{-1} . (It is a property of unimodular (e.g. abelian) groups that this does not affect the value of the integral.) Thus

$$I_\delta(s)_{\text{fin}} = \int_{\mathbf{A}_{\text{fin}}^*} \int_{\mathbf{A}_{\text{fin}}} f_{\text{fin}}\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix} \delta\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \overline{\theta_{\text{fin}}(rx)} dx \chi^*(y) |y|_{\text{fin}}^{k/2-s} d^*y.$$

Looking at determinants, by (5.4) the integrand is nonzero only if $y = \frac{nu}{\ell^2}$ for some $\ell \in \mathbf{Q}^+$ and $u \in \widehat{\mathbf{Z}}^*$. Thus the above is

$$= \sum_{\ell \in \mathbf{Q}^+} \left(\frac{n}{\ell^2}\right)^{s-k/2} \int_{\widehat{\mathbf{Z}}^*} \int_{\mathbf{A}_{\text{fin}}} f_{\text{fin}}\left(\begin{pmatrix} \ell & \\ & 1 \end{pmatrix}^{-1} \begin{pmatrix} nu & \\ & \ell \end{pmatrix} \delta\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \overline{\theta_{\text{fin}}(rx)} dx \chi^*\left(\frac{nu}{\ell^2}\right) d^*u.$$

Note that $\chi_{\text{fin}}^*\left(\frac{n}{\ell^2}\right) = \chi^*\left(\frac{n}{\ell^2}\right) = 1$ since $\frac{n}{\ell^2} \in \mathbf{Q}^+$. Likewise, the scalar factor of ℓ_{fin}^{-1} pulls out of f_{fin} as $\psi_{\text{fin}}^*(\ell^{-1}) = 1$. Hence

$$(7.5) \quad I_\delta(s)_{\text{fin}} = \sum_{\ell \in \mathbf{Q}^+} \left(\frac{n}{\ell^2}\right)^{s-k/2} \int_{\widehat{\mathbf{Z}}^*} \int_{\mathbf{A}_{\text{fin}}} f_{\text{fin}}\left(\begin{pmatrix} nu & \\ & \ell \end{pmatrix} \delta\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \overline{\theta_{\text{fin}}(rx)} dx \chi^*(u) d^*u.$$

Proposition 7.2. *When $\delta = 1$, the integral*

$$I_1(s) = \int_{\mathbf{A}^*} \int_{\mathbf{A}} f\left(\begin{pmatrix} y & xy \\ 0 & 1 \end{pmatrix} \overline{\theta(rx)} dx \chi^*(y) |y|^{k/2-s} d^*y$$

converges absolutely on $0 < \text{Re}(s) < k-1$, and for such s it is equal to

$$(7.6) \quad \frac{n^{1-k/2}}{e^{2\pi r}} \frac{2^{k-1} (2\pi r n)^{k-s-1}}{(k-2)!} \Gamma(s) \nu(N) \sum_{d|(n,r)} d^{2s-k+1} \psi\left(\frac{n}{d}\right) \chi\left(\frac{rn}{d^2}\right).$$

Proof. The absolute convergence was proven in Proposition 7.1. We factorize the integral as $I_1(s) = I_1(s)_\infty I_1(s)_{\text{fin}}$. By (7.4) and the proof of Proposition 3.4 of [KL2], we have

$$(7.7) \quad I_1(s)_\infty = \frac{2^{k-1} (2\pi r)^{k-s-1}}{(k-2)! e^{2\pi r}} \Gamma(s).$$

Now consider the finite part, which by (7.5) is

$$I_1(s)_{\text{fin}} = \sum_{\ell \in \mathbf{Q}^+} \left(\frac{n}{\ell^2}\right)^{s-k/2} \int_{\widehat{\mathbf{Z}}^*} \int_{\mathbf{A}_{\text{fin}}} f_{\text{fin}}\left(\begin{pmatrix} nu & xnu \\ 0 & \ell \end{pmatrix}\right) \overline{\theta_{\text{fin}}(rx)} dx \chi^*(u) d^*u.$$

Replacing x by $\frac{\ell x}{nu}$, the above is

$$\begin{aligned} &= \sum_{\ell \in \mathbf{Q}^+} \frac{n^{s-k/2+1}}{\ell^{2s-k+1}} \sum_{m \in (\mathbf{Z}/D\mathbf{Z})^*} \int_{\widehat{\mathbf{Z}}^*} \int_{\mathbf{A}_{\text{fin}}} f_m\left(\begin{pmatrix} nu & x \\ 0 & \ell \end{pmatrix}\right) \overline{\theta_{\text{fin}}\left(\frac{r\ell x}{nu}\right)} dx \chi^*(u) d^*u \\ &= \sum_{\ell \in \mathbf{Q}^+} \frac{n^{s-k/2+1}}{\ell^{2s-k+1}} \sum_m \int_{\widehat{\mathbf{Z}}^*} \int_{\mathbf{A}_{\text{fin}}} f_m\left(\begin{pmatrix} 1 & -m/D \\ 0 & 1 \end{pmatrix} \begin{pmatrix} nu & x + \frac{\ell m}{D} \\ 0 & \ell \end{pmatrix}\right) \overline{\theta_{\text{fin}}\left(\frac{r\ell x}{nu}\right)} dx \chi^*(u) d^*u. \end{aligned}$$

By (5.4), the integrand is nonzero if and only if $\frac{nu}{\ell}, \ell, x + \frac{\ell m}{D} \in \widehat{\mathbf{Z}}$. Replacing x by $x - \frac{\ell m}{D}$, we obtain

$$\begin{aligned} I_1(s)_{\text{fin}} &= \frac{\nu(N)}{\tau(\overline{\chi})} \sum_{\ell|n} \frac{n^{s-k/2+1}}{\ell^{2s-k+1}} \psi^*(\ell_N) \sum_m \overline{\chi(m)} \int_{\widehat{\mathbf{Z}}^*} \int_{\widehat{\mathbf{Z}}} \overline{\theta_{\text{fin}}\left(\frac{r\ell(x - \frac{\ell m}{D})}{nu}\right)} dx \chi^*(u) d^*u \\ &= \frac{\nu(N)}{\tau(\overline{\chi})} \sum_{\ell|n} \frac{n^{s-k/2+1}}{\ell^{2s-k+1}} \psi(\ell) \sum_m \overline{\chi(m)} \int_{\widehat{\mathbf{Z}}^*} \theta_{\text{fin}}\left(\frac{r\ell^2 m}{nDu}\right) \int_{\widehat{\mathbf{Z}}} \overline{\theta_{\text{fin}}\left(\frac{r\ell x}{nu}\right)} dx \chi^*(u) d^*u. \end{aligned}$$

The integral over $\widehat{\mathbf{Z}}$ evaluates to 1 if $\frac{n}{\ell}|r$, and 0 otherwise. Setting $d = \frac{n}{\ell}$, the above is

$$\begin{aligned} &= \frac{\nu(N)}{\tau(\overline{\chi})} \sum_{d|(n,r)} \frac{n^{s-k/2+1}}{(\frac{n}{d})^{2s-k+1}} \psi\left(\frac{n}{d}\right) \sum_m \overline{\chi(m)} \int_{\widehat{\mathbf{Z}}^*} \theta_{\text{fin}}\left(\frac{rnm}{d^2 Du}\right) \chi^*(u) d^*u \\ &= \frac{\nu(N)n^{k/2-s}}{\tau(\overline{\chi})} \sum_{d|(n,r)} d^{2s-k+1} \psi\left(\frac{n}{d}\right) \sum_m \overline{\chi(m)} \int_{\widehat{\mathbf{Z}}^*} \theta_{\text{fin}}\left(\frac{rnm}{d^2 D}u\right) \overline{\chi^*(u)} d^*u \\ &= \frac{\nu(N)n^{k/2-s}}{\tau(\overline{\chi})} \sum_{d|(n,r)} d^{2s-k+1} \psi\left(\frac{n}{d}\right) \sum_m \overline{\chi(m)} \chi\left(\frac{rnm}{d^2}\right) \frac{\tau(\overline{\chi})}{\varphi(D)} \\ &= \nu(N)n^{k/2-s} \sum_{d|(n,r)} d^{2s-k+1} \psi\left(\frac{n}{d}\right) \chi\left(\frac{rn}{d^2}\right). \end{aligned}$$

Passing to the third line, we applied (2.5). Multiplying by (7.7) gives the result. \square

Although we computed the orbital integral globally, it may be of interest to know the value of the local orbital integrals, for example if one wishes to compute the analogous trace formula over a number field, or use a test function which differs from ours at a finite number of places. Letting

$$I_1(s)_p = \int_{\mathbf{Q}_p^*} \int_{\mathbf{Q}_p} f_p\left(\begin{pmatrix} y & xy \\ 0 & 1 \end{pmatrix}\right) \overline{\theta_p(rx)} \chi_p(y) |y|_p^{k/2-s} dx d^*y,$$

by calculations very similar to the above, we find, for $r \in \mathbf{Z}_p$,

$$(7.8) \quad I_1(s)_p = \begin{cases} \chi_p(r) & (p|D) \\ \nu(p^{N_p}) & (p|N) \\ (p^{n_p})^{k/2-s} \sum_{d_p=0}^{\min(r_p, n_p)} (p^{d_p})^{2s-k+1} \psi_p\left(\frac{p^{d_p}}{p^{n_p}}\right) \chi_p\left(\frac{p^{2d_p}}{p^{n_p}}\right) & (p|n) \\ 1 & (p \nmid nND). \end{cases}$$

Proposition 7.3. *When $\delta = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$, the integral*

$$I_\delta(s) = \int_{\mathbf{A}^*} \int_{\mathbf{A}} f\left(\begin{pmatrix} 0 & -y \\ 1 & x \end{pmatrix}\right) \overline{\theta(rx)} dx \chi^*(y) |y|^{k/2-s} d^*y$$

converges absolutely on $1 < \operatorname{Re}(s) < k$, and for such s it vanishes unless $N = 1$. When $N = 1$ (so k is even by (2.6)),

$$(7.9) \quad I_\delta(s) = \frac{n^{1-k/2}}{e^{2\pi r}} \frac{2^{k-1}(2\pi r n)^{s-1}}{(k-2)!} \Gamma(k-s) \frac{i^k}{D^{2s-k}} \frac{\tau(\chi)^2}{D} \sum_{d|(r,n)} d^{k-2s+1} \overline{\chi\left(\frac{rn}{d^2}\right)}.$$

Remark: Comparing with the identity term (7.6) when $N = 1$, we see that

$$\frac{i^k}{D^{2s-k}} \frac{\tau(\chi)^2}{D} I_1(k-s, \overline{\chi}) = I_\delta(s, \chi),$$

mirroring the functional equation (1.2) on the spectral side.

Proof. The absolute convergence was proven in Proposition 7.1. By (7.4) and the proof of Proposition 3.5 of [KL2], we have

$$(7.10) \quad I_\delta(s)_\infty = \frac{2^{k-1}(2\pi r)^{s-1}}{e^{2\pi r}(k-2)! i^k} \Gamma(k-s).$$

Now consider the finite part (7.5):

$$I_\delta(s)_{\text{fin}} = \sum_{\ell \in \mathbf{Q}^+} \left(\frac{n}{\ell^2}\right)^{s-k/2} \int_{\widehat{\mathbf{Z}}^*} \int_{\mathbf{A}_{\text{fin}}} f_{\text{fin}}\left(\begin{pmatrix} 0 & -\frac{nu}{\ell} \\ \ell & x\ell \end{pmatrix}\right) \overline{\theta_{\text{fin}}(rx)} dx \chi^*(u) d^*u.$$

Because the above matrix has determinant $nu \in n\widehat{\mathbf{Z}}^*$, we see from (5.4) that

$$(7.11) \quad \begin{pmatrix} 0 & -\frac{nu_p}{\ell} \\ \ell & \ell x_p \end{pmatrix} \in M(n, N)_p \quad \text{for all } p \nmid D$$

when $\ell x_p \in \mathbf{Z}_p$. Likewise, we can assume that for some $m \in (\mathbf{Z}/D\mathbf{Z})^*$,

$$(7.12) \quad \begin{pmatrix} 1 & \frac{m}{D} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\frac{nu}{\ell} \\ \ell & x\ell \end{pmatrix} = \begin{pmatrix} \frac{\ell m}{D} & -\frac{nu}{\ell} + \frac{m\ell x}{D} \\ \ell & x\ell \end{pmatrix} \in M(n, N).$$

The latter implies that $x\ell \in \widehat{\mathbf{Z}}$ and $\ell \in N\mathbf{Z}^+$. If $p|N$, then by considering the upper right entry of (7.11), we have $\operatorname{ord}_p(n) \geq \operatorname{ord}_p(\ell) \geq \operatorname{ord}_p(N) > 0$. This contradicts $(n, N) = 1$, and therefore we may assume that $N = 1$. From the upper left entry of (7.12), we see that $D|\ell$. Write $\ell = Dd$ for $d \in \mathbf{Z}^+$. Replacing x by $\frac{x}{\ell} = \frac{x}{Dd}$, the measure is scaled by $|Dd|_{\text{fin}}^{-1} = Dd$, so $I_\delta(s)_{\text{fin}}$ is equal to

$$\sum_{d \in \mathbf{Z}^+} \frac{n^{s-k/2}}{(Dd)^{2s-k-1}} \sum_{m \in (\mathbf{Z}/D\mathbf{Z})^*} \int_{\widehat{\mathbf{Z}}^*} \int_{\widehat{\mathbf{Z}}} f_m\left(\begin{pmatrix} 1 - \frac{m}{D} & \\ 0 & 1 \end{pmatrix} \begin{pmatrix} md & -\frac{nu}{Dd} + \frac{mx}{D} \\ Dd & x \end{pmatrix}\right) \overline{\theta_{\text{fin}}\left(\frac{rx}{Dd}\right)} dx \chi^*(u) d^*u.$$

From the upper right-hand entry of (7.12), $-\frac{nu}{Dd} + \frac{mx}{D} \in \widehat{\mathbf{Z}}$. Since $x \in \widehat{\mathbf{Z}}$, this means

$$(7.13) \quad mdx \in md\widehat{\mathbf{Z}} \cap (nu + Dd\widehat{\mathbf{Z}}).$$

Generally, it is not hard to show that for any $h, j, k \in \widehat{\mathbf{Z}}$,

$$(7.14) \quad h\widehat{\mathbf{Z}} \cap (j + k\widehat{\mathbf{Z}}) = \begin{cases} hc + \frac{hk}{\gcd(h, k)} \widehat{\mathbf{Z}} & \text{if } \gcd(h, k) | j \\ \emptyset & \text{if } \gcd(h, k) \nmid j, \end{cases}$$

where $c \in \mathbf{Z}$ is any solution to $hc \equiv j \pmod{k\widehat{\mathbf{Z}}}$. Applying this to (7.13), we have $\gcd(h, k) = \gcd(md, Dd) = d$, so the set in (7.13) is nonempty if and only if $d|n$. Assuming this holds, the range of x is determined by

$$x \in c_m + D\widehat{\mathbf{Z}},$$

where c_m is any positive integer satisfying

$$(7.15) \quad mdc_m \equiv nu \pmod{dD\widehat{\mathbf{Z}}}.$$

For such x , the value of f_m is identically equal to $\frac{\overline{\chi(m)}}{\tau(\overline{\chi})}$ since $N = 1$ (and so $\psi = 1$). Replace x by $c_m + Dx$. This changes the measure by a factor of $|D|_{\text{fin}} = D^{-1}$, and

$$I_\delta(s)_{\text{fin}} = \frac{n^{s-k/2}}{\tau(\overline{\chi})D^{2s-k}} \sum_{d|n} d^{k-2s+1} \sum_m \overline{\chi(m)} \int_{\widehat{\mathbf{Z}}^*} \int_{\widehat{\mathbf{Z}}} \overline{\theta_{\text{fin}}\left(\frac{r(c_m+Dx)}{Dd}\right)} dx \chi^*(u) d^*u.$$

The inner integral $\int_{\widehat{\mathbf{Z}}} \overline{\theta_{\text{fin}}\left(\frac{rx}{d}\right)} dx$ is equal to 1 if $d|r$, and 0 otherwise. Therefore the above is

$$= \frac{n^{s-k/2}}{\tau(\overline{\chi})D^{2s-k}} \sum_{d|(r,n)} d^{k-2s+1} \sum_m \overline{\chi(m)} \int_{\widehat{\mathbf{Z}}^*} \overline{\theta_{\text{fin}}\left(\frac{rc_m}{Dd}\right)} \chi^*(u) d^*u.$$

Since $d|(r, n)$, (7.15) is equivalent to $c_m \equiv \overline{m}(\frac{n}{d})u \pmod{D\widehat{\mathbf{Z}}}$, where $m\overline{m} \equiv 1 \pmod{D}$. Using this along with (2.5), we find:

$$\begin{aligned} I_\delta(s)_{\text{fin}} &= \frac{n^{s-k/2}}{\tau(\overline{\chi})D^{2s-k}} \sum_{d|(r,n)} d^{k-2s+1} \sum_m \overline{\chi(m)} \int_{\widehat{\mathbf{Z}}^*} \theta_{\text{fin}}\left(\frac{-(\frac{r}{d})(\frac{n}{d})\overline{m}u}{D}\right) \chi^*(u) d^*u \\ &= \frac{n^{s-k/2}}{\tau(\overline{\chi})D^{2s-k}} \sum_{d|(r,n)} d^{k-2s+1} \sum_m \overline{\chi(m)} \chi(-1) \overline{\chi\left(\frac{rn}{d^2}\right)} \chi(m) \frac{\tau(\chi)}{\varphi(D)} \\ (7.16) \quad &= \frac{n^{s-k/2}}{D^{2s-k}} \frac{\tau(\chi)}{\chi(-1)\tau(\overline{\chi})} \sum_{d|(r,n)} d^{k-2s+1} \overline{\chi\left(\frac{rn}{d^2}\right)} \end{aligned}$$

Since χ is primitive, we have $\tau(\chi)\chi(-1)\tau(\overline{\chi}) = \tau(\chi)\overline{\tau(\chi)} = D$. Therefore $\frac{\tau(\chi)}{\chi(-1)\tau(\overline{\chi})} = \frac{\tau(\chi)^2}{D}$. Using this and multiplying the above by (7.10), equation (7.9) follows. \square

We state here the value of the local orbital integrals

$$I_\delta(s)_p = \int_{\mathbf{Q}_p^*} \int_{\mathbf{Q}_p} f_p\left(\begin{pmatrix} 0 & -y \\ 1 & x \end{pmatrix}\right) \overline{\theta_p(rx)} \chi_p(y) |y|_p^{k/2-s} dx d^*y.$$

By local calculations similar to the above, we find, for $\delta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$,

$$(7.17) \quad I_\delta(s)_p = \begin{cases} (p^{D_p})^{k-2s} \overline{\psi_p(D)} \overline{\chi_p(D^2)} \overline{\chi_p(-r)} \frac{\tau(\chi)_p}{\tau(\overline{\chi})_p} & (p|D) \\ 0 & (p|N) \\ (p^{n_p})^{s-k/2} \sum_{d_p=0}^{\min(r_p, n_p)} (p^{d_p})^{k-2s+1} \overline{\psi_p(p^{d_p})} \chi_p\left(\frac{p^{n_p}}{p^{2d_p}}\right) & (p|n) \\ 1 & (p \nmid nND). \end{cases}$$

Here, recall that $\tau(\chi)_p = \chi_p(\frac{D}{p^{D_p}})\tau(\chi_p)$ as in (3.9). Using (3.10), it is straightforward to show that the product of the above over all p agrees with (7.16) when $N = 1$.

8. COMPUTATION OF $I_{\delta_t}(s)$

In this section we will prove the following.

Proposition 8.1. *When $\delta_t = \begin{pmatrix} t & -1 \\ 1 & 0 \end{pmatrix}$, the integral*

$$I_{\delta_t}(s) = \int_{\mathbf{A}^*} \int_{\mathbf{A}} f\left(\begin{pmatrix} yt & yxt - y \\ 1 & x \end{pmatrix}\right) \overline{\theta(rx)} dx \chi^*(y) |y|^{k/2-s} d^*y$$

converges absolutely on the strip $0 < \sigma < k$, where $\sigma = \operatorname{Re}(s)$. The integrand vanishes unless $t = \frac{Nb}{nD}$ for $b \in \mathbf{Z} - \{0\}$. When $1 < \sigma < k-1$, the sum $\sum_{t \in \mathbf{Q}^} I_{\delta_t}(s)$ is absolutely convergent, and $E := \frac{e^{2\pi r}}{\nu(N)n^{1-k/2}} \sum I_{\delta_t}(s)$ is equal to*

$$\frac{(4\pi rn)^{k-1} \varphi(D) \psi(nD) e^{i\pi s/2}}{N^s D^{s-k} (k-2)! \tau(\overline{\chi})} \sum_{\substack{a \neq 0, d > 0 \text{ sat. (8.2),} \\ \gcd(a, Nd^{(D)}) | \gcd(r, n)}} \frac{a^{s-k} \gcd(a, Nd^{(D)})}{d^s \psi(a) e^{\frac{2\pi i r \ell}{ad_D}}} J_{\chi}(a, d)_1 f_1(s; k; \frac{-2\pi i r n D}{Nad}),$$

where: $a^s = e^{-i\pi s} |a|^s$ if $a < 0$, $d^{(D)} = \prod_{p \nmid D} p^{d_p}$ is the prime-to- D part of d , and similarly for $d = d^{(D)} d_D$, ℓ is any integer satisfying $Nd^{(D)} \ell \equiv -nD \pmod{ad_D}$, J_{χ} is a product of certain explicit local factors of absolute value ≤ 1 given in (8.13),

$$(8.1) \quad {}_1f_1(s, k; w) = \frac{\Gamma(s)\Gamma(k-s)}{\Gamma(k)} {}_1F_1(s; k; w) = \int_0^1 e^{wx} x^{s-1} (1-x)^{k-s-1} dx$$

for $\operatorname{Re}(k) > \operatorname{Re}(s) > 0$, and writing a_p, d_p for the p -adic valuations of a, d , we have

$$(8.2) \quad p|D \implies \begin{cases} a_p = D_p & \text{if } d_p > D_p \\ a_p \geq D_p & \text{if } d_p = D_p \\ a_p = d_p & \text{if } 0 \leq d_p < D_p. \end{cases}$$

Remark: We give an expression for $I_{\delta_t}(s)$ in (8.16) below. As in [KL2], this can be used in principle to compute the sum over t to any level of precision.

The absolute convergence was proven in Proposition 7.1. To begin the computation, write $\delta = \delta_t$, and consider the finite part given by (7.5):

$$I_{\delta}(s)_{\text{fin}} = \sum_{\ell \in \mathbf{Q}^+} \left(\frac{n}{\ell^2}\right)^{s-k/2} \int_{\widehat{\mathbf{Z}}^*} \int_{\mathbf{A}_{\text{fin}}} f_{\text{fin}}\left(\begin{pmatrix} \frac{nut}{\ell} & \frac{nux}{\ell} - \frac{nu}{\ell} \\ \ell & \ell x \end{pmatrix}\right) \overline{\theta_{\text{fin}}(rx)} dx \chi^*(u) d^*u.$$

We will show that this vanishes unless $t \in \frac{N}{nD} \mathbf{Z}$. In anticipation of this, write $t = \frac{Nb}{nD}$, where (for now) $b \in \mathbf{Q}^*$. Since the determinant of the matrix is $nu \in n\widehat{\mathbf{Z}}^*$, by (5.4) the integrand vanishes unless $\ell \in n\mathbf{Z}^+$ and $\ell x \in \widehat{\mathbf{Z}}$. Therefore we shall set $\ell = Nd$, and replace x by $\ell^{-1}x = x/Nd$, so dx becomes $d(x/Nd) = |Nd|_{\text{fin}}^{-1} dx = Nd \cdot dx$. The above then becomes

$$(8.3) \quad \sum_{d \in \mathbf{Z}^+} \frac{n^{s-k/2} Nd}{(Nd)^{2s-k}} \int_{\widehat{\mathbf{Z}}^*} \int_{\widehat{\mathbf{Z}}} f_{\text{fin}}\left(\begin{pmatrix} \frac{ub}{dD} & \frac{ubx}{Nd^2D} - \frac{nu}{Nd} \\ Nd & x \end{pmatrix}\right) \overline{\theta_{\text{fin}}\left(\frac{rx}{Nd}\right)} dx \chi^*(u) d^*u.$$

We will show that the integrand vanishes unless $b \in \mathbf{Z}$ and $d|b$. We now work locally. The p -th factor of the double integral is

$$(8.4) \quad \int_{\mathbf{Z}_p^*} \int_{\mathbf{Z}_p} f_p\left(\begin{pmatrix} \frac{ub}{dD} & \frac{ubx}{Nd^2D} - \frac{nu}{Nd} \\ Nd & x \end{pmatrix}\right) \overline{\theta_p\left(\frac{rx}{Nd}\right)} dx \chi_p(u) d^*u.$$

8.1. Local computation at $p|D$. Suppose first that $p|D$. Then N is a unit, so replacing u by Nu , (8.4) becomes

$$(8.5) \quad \int_{\mathbf{Z}_p^*} \int_{\mathbf{Z}_p} f_p \left(\begin{pmatrix} \frac{Nub}{dD} & \frac{ubx}{d^2D} - \frac{nu}{d} \\ Nd & x \end{pmatrix} \right) \overline{\theta_p \left(\frac{rx}{Nd} \right)} dx \chi_p(Nu) d^*u.$$

Recall that $f_p = \sum_{m \in (\mathbf{Z}_p/D\mathbf{Z}_p)^*} f_{p,m}$, where $f_{p,m} = f_{p,m}^\chi$ is given in (3.12). Fix m and consider

$$f_{p,m} \left(\begin{pmatrix} \frac{Nub}{dD} & \frac{ubx}{d^2D} - \frac{nu}{d} \\ Nd & x \end{pmatrix} \right) = f_{p,m} \left(\begin{pmatrix} 1 & -\frac{m}{D} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{Nub}{dD} + \frac{Ndm}{D} & \frac{ubx}{d^2D} - \frac{nu}{d} + \frac{mx}{D} \\ Nd & x \end{pmatrix} \right).$$

By (3.11) and the fact that the determinant of the rightmost matrix is $nNu \in \mathbf{Z}_p^*$, this is nonzero if and only if the rightmost matrix belongs to K_p , or equivalently:

- (i) $\frac{ub}{dD} + \frac{dm}{D} \in \mathbf{Z}_p$
- (ii) $\frac{x}{d} \left(\frac{ub}{dD} + \frac{dm}{D} \right) - \frac{nu}{d} \in \mathbf{Z}_p$.

We assume henceforth that these conditions hold. Notice that if $p \nmid d$, the first condition already implies the second. On the other hand, since $x \in \mathbf{Z}_p$,

$$p|d, (i), (ii) \implies \begin{cases} (iii) & x \in \mathbf{Z}_p^* \\ (iv) & \left(\frac{ub}{dD} + \frac{dm}{D} \right) \in \mathbf{Z}_p^*. \end{cases}$$

Letting $D_p = \text{ord}_p(D)$, and similarly for b_p, d_p , we find by condition (i) (if $d_p = 0$) and condition (iv) (if $d_p > 0$) that

$$(8.6) \quad p|D \implies \begin{cases} b_p = d_p + D_p & \text{if } d_p > D_p \\ b_p \geq 2D_p & \text{if } d_p = D_p \\ b_p = 2d_p & \text{if } 0 \leq d_p < D_p. \end{cases}$$

Suppose first that $p \nmid d$, so that by (8.6), $d_p = b_p = 0$. Then condition (i) is equivalent to

$$m \equiv \frac{-bu}{d^2} \pmod{D\mathbf{Z}_p},$$

d, b, u, m being units. So given u , there is exactly one m for which the above condition holds, and by (3.12) the inner integral of (8.5) is equal to

$$\int_{\mathbf{Z}_p} \frac{\overline{\chi_p(m)}}{\tau(\overline{\chi})_p} \overline{\theta_p \left(\frac{rx}{Nd} \right)} dx = \frac{\overline{\chi_p \left(\frac{-b}{d^2} \right)}}{\tau(\overline{\chi}_p)} \overline{\chi_p(u)}$$

since $\theta_p \left(\frac{rx}{Nd} \right) = 1$. Therefore the double integral (8.4) is equal to

$$(8.7) \quad \chi_p(N) \frac{\chi_p \left(\frac{-d}{b/d} \right)}{\tau(\overline{\chi})_p} \int_{\mathbf{Z}_p^*} \overline{\chi_p(u)} \chi_p(u) d^*u = \frac{\chi_p \left(\frac{-Nd}{b/d} \right)}{\tau(\overline{\chi})_p}.$$

Now suppose $p|d$. In view of (ii) and (iv), we have

$$x \in \frac{Dnu}{u \frac{b}{d} + dm} + d\mathbf{Z}_p \subset \mathbf{Z}_p^*.$$

This is the only condition on x required for the $f_{p,m}$ -term to be nonzero. Make the substitution $x = \frac{Dnu}{ub/d + dm} + d \cdot w$, so $dx = |d|_p dw = |Nd|_p dw$. The value of $f_{p,m}$ is

$\frac{\overline{\chi_p(m)}}{\tau(\overline{\chi})_p}$, so assuming (iv) holds, the inner integral in (8.5) is equal to

$$|Nd|_p \sum_m \frac{\overline{\chi_p(m)}}{\tau(\overline{\chi})_p} \overline{\theta_p\left(\frac{rDnu}{Nub+Nd^2m}\right)} \int_{\mathbf{Z}_p} \overline{\theta_p\left(\frac{rw}{N}\right)} dw.$$

The latter integral has value 1 since $r \in \mathbf{Z}^+$ and N is a unit. The variable u ranges through the set $U_{b,d,m} = \left(-\frac{d^2m}{b} + \frac{dD}{b}\mathbf{Z}_p^*\right) \cap \mathbf{Z}_p^*$ determined by condition (iv) above. By considering the possibilities for $d_p > 0$ listed in (8.6), we find easily that

$$U_{b,d,m} = \begin{cases} \mathbf{Z}_p^* & \text{if } d_p > D_p \text{ (so } b_p = d_p + D_p) \\ -\frac{d^2m}{b} + \frac{dD}{b}\mathbf{Z}_p^* & \text{if } 0 < d_p < D_p \text{ (so } b_p = 2d_p) \\ \mathbf{Z}_p^* & \text{if } d_p = D_p \text{ and } b_p > 2D_p \\ \bigcup_{\substack{a \in (\mathbf{Z}/p\mathbf{Z})^*, \\ a \not\equiv -\frac{d^2m}{b} \pmod{p}}} (a + p\mathbf{Z}_p) & \text{if } d_p = D_p \text{ and } b_p = 2D_p. \end{cases}$$

The double integral (8.4) is equal to

$$\frac{|Nd|_p}{\tau(\overline{\chi})_p} \sum_{m \in (\mathbf{Z}_p/D\mathbf{Z}_p)^*} \overline{\chi_p(m)} \int_{U_{b,d,m}} \chi_p(Nu) \overline{\theta_p\left(\frac{rDnu}{Nub+Nd^2m}\right)} d^*u.$$

Noting that $U_{b,d,m} = mU_{b,d,1}$, we can replace u by mu and integrate over $U_{b,d,1}$. This has the effect of cancelling every m , so that the above is

$$(8.8) \quad \frac{|Nd|_p \varphi_p(D)}{\tau(\overline{\chi})_p} \int_{U_{b,d,1}} \chi_p(Nu) \overline{\theta_p\left(\frac{rDnu}{Nub+Nd^2}\right)} d^*u.$$

We leave this as something that could be computed given χ_p , if desired. For our purposes, it will be enough to bound the integral trivially by 1.

8.2. Local computation at $p \nmid D$. Now we suppose $p \nmid D$. In this case, χ_p is unramified, so (8.4) is equal to

$$(8.9) \quad \int_{\mathbf{Z}_p^*} \int_{\mathbf{Z}_p} f_p\left(\left(\frac{ub}{dD} - \frac{ubx}{Nd^2D} - \frac{nu}{Nd}\right)\right) \overline{\theta_p\left(\frac{rx}{Nd}\right)} dx d^*u.$$

Since the support of f_p is $Z(\mathbf{Q}_p)M(n, N)_p$ and the determinant of the above matrix is $nu \in n\mathbf{Z}_p^*$, the integrand is nonzero precisely when

- (i) $\frac{ub}{dD} \in \mathbf{Z}_p$
- (ii) $\frac{x}{Nd}\left(\frac{ub}{dD}\right) - \frac{nu}{Nd} \in \mathbf{Z}_p.$

Both conditions are in fact independent of u . By (i), we see that $0 \leq d_p \leq b_p$ since $p \nmid D$. Together with (8.6), this proves our assertion that $I_\delta(s)$ vanishes unless $b \in \mathbf{Z}$, and that the sum in (8.3) can be taken just over $d|b$. Since $u, D \in \mathbf{Z}_p^*$, condition (ii) is equivalent to

$$(8.10) \quad \frac{b}{d}x \in (Dn + Nd\mathbf{Z}_p) \cap \frac{b}{d}\mathbf{Z}_p.$$

(If $p|N$, this is possible only if $d_p = b_p$.) Applying the local analog of (7.14) to (8.10), and then dividing by $\frac{b}{d}$, we find that

$$x \in \begin{cases} c + \frac{Nd}{\gcd(b/d, Nd)}\mathbf{Z}_p & \text{if } \gcd_p(b/d, Nd) | Dn \\ \emptyset & \text{otherwise,} \end{cases}$$

where \gcd_p denotes the p -part of the gcd, and $c \in \mathbf{Z}$ is given by

$$\frac{b}{d}c \equiv Dn \pmod{Nd\mathbf{Z}_p}.$$

We shall specify c further as follows, so that the above holds simultaneously for all $p \nmid D$. Write $d = d^{(D)}d_D$, where $(D, d^{(D)}) = 1$ and $d_D = \prod_{p|D} p^{d_p}$. Then we take $c \in \mathbf{Z}$ so that:

$$(8.11) \quad \begin{cases} \frac{b}{d}c \equiv Dn \pmod{Nd^{(D)}\mathbf{Z}} \\ c \equiv 0 \pmod{d_D\mathbf{Z}}. \end{cases}$$

It is not hard to see that such c exists under the hypothesis $\gcd_p(b/d, Nd) | Dn$ for all $p \nmid D$. Indeed, $\prod_{p \nmid D} \gcd_p(\frac{b}{d}, Nd) = \gcd(\frac{b}{d}, Nd^{(D)}) | Dn$, which implies the existence of an integer c satisfying the first congruence. If necessary we can multiply c by $d_D \overline{d_D} \equiv 1 \pmod{Nd^{(D)}}$ to further ensure that $c \in d_D\mathbf{Z}$.

The first congruence in (8.11) implies that b/d is relatively prime to N . Therefore $\psi_p(x) = \psi_p(c)$. Since $\text{meas}(\mathbf{Z}_p^*) = 1$ and everything is independent of $u \in \mathbf{Z}_p^*$, the double integral (8.9) is equal to

$$\nu_p(N)\psi_p((cN)_p) \int_{c + \frac{Nd}{\gcd(b/d, Nd)}\mathbf{Z}_p} \overline{\theta_p(\frac{rx}{Nd})} dx,$$

where we used the formula (4.1) for f_p . Now let $x = c + \frac{Nd}{\gcd(b/d, Nd)}w$. Then the above is

$$(8.12) \quad = \nu_p(N)\psi_p((cN)_p) \left| \frac{Nd}{\gcd(b/d, Nd)} \right|_p \overline{\theta_p(\frac{rc}{Nd})} \int_{\mathbf{Z}_p} \overline{\theta_p(\frac{rw}{\gcd(b/d, Nd)})} dw.$$

The integral is nonzero (hence equal to 1) if and only if $\gcd_p(b/d, Nd) | r$.

8.3. The finite part. Multiply the local factors (8.8) (resp. (8.7)) and (8.12), together with the coefficient of the double integral in (8.3). We set

$$(8.13) \quad J_\chi(\frac{b}{d}, d) = \prod_{p|D} J_p(\frac{b}{d}, d),$$

where $J_p(\frac{b}{d}, d)$ denotes the integral in (8.8) if $d_p > 0$ (resp. the quantity $\frac{\chi_p(\frac{-Nd}{b/d})}{\varphi_p(D)}$ if $d_p = 0$). We find:

$$\begin{aligned} I_{\delta_t}(s)_{\text{fin}} &= \sum_{\substack{d|b \text{ satisfying (8.6),} \\ \gcd(\frac{b}{d}, Nd^{(D)}) | (r, n)}} \frac{n^{s-k/2} Nd}{(Nd)^{2s-k}} J_\chi(\frac{b}{d}, d) \prod_{p|D} \frac{|Nd|_p \varphi_p(D)}{\tau(\overline{\chi})_p} \\ &\quad \times \prod_{p \nmid D} \nu_p(N)\psi_p((cN)_p) \left| \frac{Nd}{\gcd(\frac{b}{d}, Nd)} \right|_p \overline{\theta_p(\frac{rc}{Nd})}. \end{aligned}$$

Here $d^{(D)} = \prod_{p \nmid D} p^{d_p}$ as before. We can make a few simplifications. First,

$$\prod_{p \nmid D} \psi_p((cN)_p) = \psi^*(cN) = \psi(c) = \frac{\psi(nD)}{\psi(b/d)}$$

since $\frac{b}{d}c \equiv nD \pmod{N}$ and $(\frac{b}{d}, N) = 1$ by (8.10). By the second congruence of (8.11), namely $d_D | c$, we have $\theta_p(\frac{rc}{Nd}) = 1$ for all $p|D$. Hence

$$\prod_{p \nmid D} \overline{\theta_p(\frac{rc}{Nd})} = \overline{\theta_{\text{fin}}(\frac{rc}{Nd})} = \theta_\infty(\frac{rc}{Nd}) = e^{-\frac{2\pi i rc}{Nd}}.$$

Therefore

$$(8.14) \quad I_{\delta_t}(s)_{\text{fin}} = \frac{n^{s-k/2} \varphi(D) \nu(N)}{N^{2s-k} \tau(\bar{\chi})} \sum_{\substack{d|b \text{ satisfying (8.6),} \\ \gcd(\frac{b}{d}, Nd^{(D)})|(r,n)}} \frac{\psi(nD) \gcd(\frac{b}{d}, Nd^{(D)})}{\psi(\frac{b}{d}) d^{2s-k} e^{\frac{2\pi i r c}{Nd}}} J_{\chi}(\frac{b}{d}, d).$$

8.4. Archimedean integral and global expression. Finally, we consider the archimedean integral $I_{\delta_t}(s)_{\infty}$. By (7.4) and the proof of Proposition 3.7 of [KL2], we have

$$I_{\delta_t}(s)_{\infty} = \frac{(4\pi r)^{k-1} t^{\bar{s}-k}}{(k-2)! e^{2\pi r}} e^{-i\pi \bar{s}/2} e^{-2\pi i r/t} {}_1f_1(\bar{s}; k; 2\pi i r/t),$$

where $t^s = e^{i\pi s}|t|^s$ if $t < 0$. Therefore

$$I_{\delta_t}(s)_{\infty} = \frac{(4\pi r)^{k-1} t^{s-k} e^{i\pi s/2} e^{2\pi i r/t}}{(k-2)! e^{2\pi r}} {}_1f_1(s; k; -2\pi i r/t),$$

where now $t^s = e^{-i\pi s}|t|^s$ if $t < 0$. By the discussion above, we can take $t = Nb/nD$, so

$$(8.15) \quad I_{\delta_t}(s)_{\infty} = \frac{(4\pi r)^{k-1} N^{s-k} e^{i\pi s/2}}{(k-2)! e^{2\pi r} n^{s-k} D^{s-k}} b^{s-k} e^{\frac{2\pi i r n D}{Nb}} {}_1f_1(s; k; \frac{-2\pi i r n D}{Nb}).$$

When we multiply by the finite part (8.14), the terms $e^{2\pi i r n D/Nb}$ and $e^{-2\pi i r c/Nd}$ combine as follows. By (8.11) we can write $(\frac{b}{d})c = nD + \ell Nd^{(D)}$, where $\ell \in \mathbf{Z}$ is an integer satisfying

$$\ell Nd^{(D)} \equiv -nD \pmod{(\frac{b}{d})d_D}.$$

Conversely, any ℓ satisfying the above determines an integer c satisfying (8.11). Then

$$e^{\frac{2\pi i r n D}{Nb}} e^{-\frac{2\pi i r c}{Nd}} = e^{\frac{2\pi i r (nD - \frac{b}{d}c)}{Nb}} = e^{\frac{-2\pi i r \ell Nd^{(D)}}{Nb}} = e^{\frac{-2\pi i r \ell}{(b/d)d_D}}.$$

Multiplying (8.15) by the finite part (8.14), we find, for $t = \frac{Nb}{nD}$,

$$(8.16) \quad \begin{aligned} I_{\delta_t}(s) &= \frac{(4\pi r)^{k-1} N^{s-k} e^{i\pi s/2}}{(k-2)! e^{2\pi r} n^{s-k} D^{s-k}} b^{s-k} {}_1f_1(s; k; \frac{-2\pi i r n D}{Nb}) \\ &\times \frac{n^{s-k/2} \varphi(D) \nu(N)}{N^{2s-k} \tau(\bar{\chi})} \sum_{\substack{d|b \text{ sat. (8.6),} \\ \gcd(\frac{b}{d}, Nd^{(D)})|(r,n)}} \frac{\psi(nD) \gcd(\frac{b}{d}, Nd^{(D)})}{\psi(\frac{b}{d}) d^{2s-k} e^{\frac{2\pi i r \ell}{(b/d)d_D}}} J_{\chi}(\frac{b}{d}, d). \end{aligned}$$

Writing $b = ad$, the condition (8.6) becomes (8.2). Summing over t , we see that

$$\frac{e^{2\pi r}}{\nu(N)n^{1-\frac{k}{2}}} \sum_{t \in \mathbf{Q}^*} I_{\delta_t}(s) \text{ is equal to } \frac{(4\pi r n)^{k-1} \varphi(D) \psi(nD) e^{i\pi s/2}}{N^s D^{s-k} (k-2)! \tau(\bar{\chi})} \sum_{\substack{a \neq 0, d > 0 \text{ sat. (8.2),} \\ \gcd(a, Nd^{(D)}) | \gcd(r, n)}} \frac{a^{s-k} \gcd(a, Nd^{(D)})}{d^s \psi(a) e^{\frac{2\pi i r \ell}{ad_D}}} J_{\chi}(a, d) {}_1f_1(s; k; \frac{-2\pi i r n D}{Nad}),$$

where $a^s = e^{-i\pi s}|a|^s$ if $a < 0$. This completes the proof of Proposition 8.1.

9. ASYMPTOTICS

Grouping a with $-a$, we rewrite the above sum as follows:

$$\sum_{\substack{a, d > 0 \text{ sat. (8.2),} \\ \gcd(a, Nd^{(D)}) | \gcd(r, n)}} \left[\frac{a^{s-k}}{\psi(a)e^{\frac{2\pi i r \ell}{adD}}} J_\chi(a, d)_1 f_1(s; k; \frac{-2\pi i r n D}{Nad}) \right. \\ \left. + \frac{e^{-i\pi s}(-1)^k a^{s-k}}{\psi(-1)\psi(a)e^{\frac{-2\pi i r \ell}{adD}}} J_\chi(-a, d)_1 f_1(s; k; \frac{2\pi i r n D}{Nad}) \right] \frac{\gcd(a, Nd^{(D)})}{d^s}.$$

From the integral representation (8.1), we see that

$$(9.1) \quad |{}_1 f_1(s, k, 2\pi i w)| \leq B(\sigma, k - \sigma) \leq 1$$

when $1 \leq \sigma \leq k - 1$, where $B(x, y) = \int_0^1 u^{x-1}(1-u)^{y-1} du = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the Beta function. Because $|J_\chi(a, d)| \leq 1$, the absolute value of the above is

$$\leq \gcd(r, n) B(\sigma, k - \sigma) (1 + e^{\pi\tau}) \sum_{a, d > 0} a^{\sigma-k} d^{-\sigma} \quad (s = \sigma + i\tau).$$

Using $|e^{i\pi s/2}|(1 + e^{\pi\tau}) = 2 \cosh(\pi\tau/2)$, we obtain the following proposition.

Proposition 9.1. *Write $s = \sigma + i\tau$ for $1 < \sigma < k - 1$. Then the term E given in Proposition 8.1 satisfies the bound*

$$|E| \leq \frac{(4\pi r n)^{k-1} D^{k-\sigma-\frac{1}{2}} \varphi(D) \gcd(r, n) B(\sigma, k - \sigma)}{N^\sigma (k-2)!} 2 \cosh(\frac{\pi\tau}{2}) \zeta(k - \sigma) \zeta(\sigma).$$

Theorem 1.1 now follows immediately. In order to prove Corollary 1.2, we must show that the quotient $Q = \frac{E}{F}$ has the limit 0 as $N + k \rightarrow \infty$, where F is the first geometric term of (1.4), and E is the error term of (1.4) discussed above. We take $k \geq 3$, $N > 1$ and $\gcd(n, r) = 1$, so for $\frac{k-1}{2} < \sigma < \frac{k+1}{2}$, we have $|F| = \frac{2^{k-1} (2\pi r n)^{k-\sigma-1} |\Gamma(s)|}{(k-2)!}$. Thus by the above proposition and (9.1),

$$(9.2) \quad |Q| = \left| \frac{E}{F} \right| \ll_{D, \tau} \frac{D^{k-\sigma} (2\pi r n)^\sigma}{N^\sigma |\Gamma(s)|} \zeta(k - \sigma) \zeta(\sigma).$$

We write $\sigma = \frac{k}{2} + \delta$ for $|\delta| < \frac{1}{2}$. Then each zeta factor is bounded by the constant $\zeta(\frac{3}{2} - \delta)$. By Stirling's approximation ([AS], 6.1.39),

$$\Gamma(s)^{-1} = \Gamma(\frac{k}{2} + \delta + i\tau)^{-1} \sim \frac{e^{k/2}}{\sqrt{2\pi} (k/2)^{\frac{k}{2} + \delta + i\tau - \frac{1}{2}}}$$

as $k \rightarrow \infty$. With (9.2), this shows that

$$|Q| \ll \frac{(4D\pi r n e)^{k/2}}{N^{\frac{k-1}{2}} k^{\frac{k}{2}-1}}$$

where the implied constant depends on δ, D, r, n, τ . This clearly goes to 0 as $N + k \rightarrow \infty$.

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